Optimal Taxation with Persistent Idiosyncratic Investment Risk

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This Draft: August 2015

Abstract

We study the role of capital taxation in a general equilibrium heterogeneous agent economy with uninsurable investment risk characterized by persistent productivity shocks to the firms privately owned by the entrepreneurs. In contrast to models with i.i.d. investment risk, the inclusion of persistent investment risk does not allow for easy aggregation. Instead, in solving this model we use a novel application of perturbation theory to economies with heterogeneous agents. This application exploits a symmetry that reduces the computational burden of approximating policy rules via a truncated Taylor expansion, allowing us to change the point of approximation to be the closest non-degenerate steady state to the current state of the planners problem as the economy evolves. The presence of idiosyncratic investment risk leads to an under-accumulation of capital, which leads the Ramsey planner to have two conflicting motives in setting the capital tax. On the one hand, the planner wishes to correct the under-accumulation of capital by subsidizing saving. On the other hand, the planner wants to use capital taxes to redistribute from agents whose investments paid off to those whose investments did not perform well. In the presence of i.i.d. productivity shocks the first motive dominates and the planner moves to correct the savings decisions of the agents, but we find that with persistent shocks the motive for redistribution dominates, leading the government to tax capital. This result is robust to the inclusion of financial frictions.

1 Introduction

This paper represents a normative exploration of the optimal fiscal policy in a neo-classical growth model where agents face idiosyncratic investment risk in the form of persistent productivity shocks to their privately owned firms. The study of optimal taxation in the presence of such risk is worthwhile because it is an important source of market incompleteness, affecting the investment decisions of entrepreneurs, managers, and privately owned businesses. In addition the presence of such risk allows models to match concentrations of wealth seen in the data. The nature of uninsurable investment risk also provides a natural motivation for the government's use of capital income taxation as it directly affects the source of income risk.

In deciding on the optimal tax levels the government faces, at times, conflicting tradeoffs. On the one hand, the government can use capital taxes, in combination with lump sum transfers, to help insure agents against shocks to the productivities of their firm. At the same time, as was present in Angeletos (2007), the presence of uninsurable investment risk can lead to the under-accumulation of capital. The Ramsey planner will want to use capital taxation to correct the under-saving of the entrepreneurs, which for lower levels would encourage the government to subsidize capital income.

This paper adapts the framework presented in Angeletos (2007), in which he presents a neoclassical growth model with agents who face uninsurable investment risk and studies its effects on macroeconomic aggregates. Specifically, we adapt his model of idiosyncratic investment risk to allow for persistent productivity shocks and include the presence of a Ramsey planner who adjusts tax rates in order to maximize welfare. The i.i.d. productivity shocks present in Angeletos (2007) allowed for an aggregation which removed the distribution of assets as a state variable of the economy. The presence of persistent risk breaks that aggregation and requires a new approach to approximate the optimal Ramsey allocation.

We approach this problem with a new method for applying perturbation theory to models with heterogeneous agents and both idiosyncratic and aggregate risk.[1] The approach features two novel departures from the standard use of perturbation theory. Ordinarily, when applying perturbation theory to models featuring ex-post heterogeneity, the policy rules are approximated around the non-stochastic steady state with a degenerate distribution of individual state variables. As the economy evolves, this implies that the approximation of the policy rules will get progressively worse. In approximating a canonical consumption-savings model these errors tend to be small[2], but a Ramsey taxation problem will include many higher order terms involving the individual state variables.

A common feature of models with uninsurable idiosyncratic risk is that there are multiple non-stochastic steady states, each associated with a different distribution of individual states. Each one of these steady states is a valid point around which to approximate the agent and

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[1] There is no aggregate risk in the baseline version of our model but we do explore extensions with aggregate risk.

[2] The savings rules in these models tend to be close to linear.

planner policy rules. By changing the point of approximation each period to be the current distribution of individual states we maintain the same level of accuracy throughout the simulation. This allows the coefficients of the truncated Taylor expansion to depend not only on the current distribution of individual state variables, but also on the current state of the agent.

While changing the point of approximation may appear natural it is associated with significant technical hurdles. Even with first order approximations the number of derivatives that need to be computed grows geometrically with the number of agents. Without exploiting some symmetry of the problem to reduce this number, using perturbation theory would be technically impossible. Ordinarily, this is done by approximating around a degenerate distribution and, as agents are identical at the non-stochastic steady state, this means that many of the derivatives will be identical. Clearly, this symmetry cannot be exploited when approximating around a non-degenerate steady state, so we instead exploit a symmetry in the evolution of the individual state variables. This allows us to decompose complicated cross derivatives into simpler terms involving the response of aggregate control variables.

An important feature of this method is its generality. The assumption required to invoke the symmetry used in this paper is much weaker than all agents being ex-ante identical. As such it can be applied to models in which agents have permanent types, such as differing in preferences. This makes it a flexible algorithm that can be applied to a range of heterogenous agent economies, allowing us to explore extensions with labor income risk, financial frictions, and aggregate shocks.

The algorithm we develop allows us to focus on how persistence of the productivity process affects long run tax rate chosen by a time zero Ramsey planner. When choosing the optimal capital tax rate the planner is balancing two competing forces. The first is correcting for the savings decisions that are distorted when entrepreneurs face uninsurable investment risk. The planner weighs this against his desire to provide ex-ante insurance/ ex-post redistribution. Numerically we find that when the productivity shocks are i.i.d. the redistributive motive is minimized and the planner moves to correct the distortions in the agents' savings decisions. We find, however, that as the persistence of the productivity shocks increases the redistributive motive dominates. Persistence links productivities across periods. Instituting a tax at time t

will allow the planner not only to redistribute from agents who had high productivity shocks in period t, but also from agents who had high productivity shocks in previous periods, thereby increasing the redistributionary/insurance value of capital taxation. The remainder of the paper is organized as follows: Section 2 outlines the environment, Section 3 presents the planner's optimal taxation problem, Section 4 gives an overview of how we apply perturbation theory to heterogeneous agent economies, Section 5 describes our results with persistent investment risk, Section 6 explores extensions to the problem with aggregateshocks and Section 7 concludes.

1.1 Related Literature

This paper is most similar to Panousi and Reis (2014). They solve for the optimal steady state capital tax in a continuous time variant of Angeletos (2007), and find that in the long run steady state the planner moves to maximize the steady state human wealth. Essentially the planner moves, as best he can, to correct the savings decisions towards first best. As the model is in continuous time with i.i.d. shocks the planner does not worry about insuring contemporaneous shocks. The planner does care about how taxes affect the distribution of consumption, but this is best handled by higher taxes earlier in the planner's problem.

The role that investment income risk should play in the choice of optimal capital is important because such risk is empirically relevant. First, using data from IRS tax returns, DeBacker et al. (2012) show that entrepreneurs face significant and persistent risk in their business over time. Secondly, Moskowitz and Vissing-Jorgensen (2002) document, using the survey consumer finances, that the private equity market is a signifiant portion of the economy. Moreover, those who own private equity are heavily exposed to the risk. Approximately 75% of private equity is owned by households with 50% of their net worth in private equity. Among those with positive private equity, an average of 80% of their private equity is held in one actively managed firm.

This paper is also related to the large literature on optimal Ramsey taxation. A famous result in that literature is the zero capital tax in the steady state in Chamley (1986) and Judd (1985). This result was generalized in Atkeson et al. (1999) to show that it holds in a wide class of models. Other papers such as Chari et al. (1994) have found this result in models with aggregate risk. In contrast, with incomplete markets the optimal capital tax tends to be nonzero. This was first shown by Aiyagari (1995), who finds that the long run optimal capital tax is positive. This result is driven by the planner attempting to correct the over-accumulation. A similar result was obtained by Park (2013), who studied a Ramsey taxation problem with limited commitment. The positive steady state capital tax in that paper was chosen to correct an externality associated with the default option. A common theme, as in Panousi and Reis (2014), is that in the long run the planner does not care about using capital taxes for redistribution, but instead moves to correct the savings decisions of agents generated by the market incompleteness.

While the Mirrleesian taxation literature primarily focuses on labor income risk, a few papers study problems with capital income risk. Albanesi (2011) studies a two period model, in which entrepreneurs can expend effort to affect the probability of success of their investment. As the effort level is private information, she finds that a negative wedge on the returns to risky capital is possible since it relaxes the incentive constraint. In Shourideh (2012), rather than having private effort, entrepreneurs are heterogeneous in ex-ante productivity and have private information regarding their own investment and consumption, but face additional risk when investing in their project. He studies a dynamic economy, and finds the presence of a bequest subsidy, as additional consumption in every state of the world relaxes the incentive compatibility of the agents. Both of these papers have the advantage of not placing ad-hoc restrictions on the tax instruments available to the planner, while the advantage of studying a Ramsey taxation problem is that it is not necessary to place the same stringent information restriction.

This paper is also related to a literature applying perturbation theory to economies with heterogeneous agents. Preston and Roca (2007) and Mertens and Judd (2013) both use use perturbation methods to approximate incomplete markets economies where agents face uninsurable income risk. Both approximate around a non-stochastic steady state with a degenerate distribution of state variables but differ in how they simplify the problem. Preston and Roca (2007) note that agents will respond only to first and second order moments of the state variables when performing a quadratic approximation, while Mertens and Judd (2013) exploit the symmetry that all agents are identical in order to reduce the number of derivatives that need to be computed. An alternative perturbation approach is provided by Reiter (2009), who performs a first order approximation around the steady state of a Bewley/Aiyagari model to approximate a model with aggregate risk. To my knowledge there is no paper which uses perturbation theory and changes the point of approximation while simulating the model.

2 Environment

We study a discrete time, infinite horizon economy with a continuum of ex-ante identical agents. These agents are uniformly distributed across the unit interval $[0,1]$. Each agent is both an entrepreneur and a worker supplying labor to a global labor market. As an entrepreneur the agent invests and runs his own private firm. Each period, given the current private capital stock and productivity, the agent makes hiring choices and chooses next period's capital level. The firm can only use capital invested by that particular entrepreneur. The agents face persistent idiosyncratic shocks to the productivity of their own firm, which they can only partially insure against through a risk free bond. The government has access to proportional taxes on both labor and capital income as well as lump sum transfers. It uses these instruments to redistribute resources across agents.

2.1 Firms

Each agent i runs a firm with production function

$$f(k^i_{t-1}, n^i_t, A^i_t) = A^i_t (k^i_{t-1})^{\xi_k} (n^i_t)^{\xi_l} \tag{1}$$

where k^i_{t-1} is the capital invested in period $t-1$, and n^i_t is the amount of labor hired this period. Capital depreciates at a rate δ. For computational tractability we assume that $\xi_k + \xi_l < 1$, and therefore the firms face decreasing returns to scale.[3] The capital in the production function was chosen in the previous period and cannot be changed, and therefore the entrepreneur's sole problem is to choose the amount of labor to hire. The entrepreneur solves

$$\max_{n^i_t} f(k^i_{t-1}, n^i_t, A^i_t) - W_t n^i_t, \tag{2}$$

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[3] Our computational method for approximating the Ramsey plan relies on taking the limit as the size of the shocks approaches zero. With heterogeneous productivities and constant returns to scale, all of the capital would be held by the most productive firm.

where W_t is the global wage rate. The first order condition with respect to n_t^i gives

$$W_t = A_t^i \xi_l (k_{t-1}^i)^{\xi_k} (n_t^i)^{\xi_l - 1}. \tag{3}$$

Therefore, $\pi_t^i = \pi(k_{t-1}^i, A_t^i | W_t)$ and $n_t^i = n(k_{t-1}^i, A_t^i | W_t)$ with

$$n(k_{-}, A|W) = (A\xi_l)^{\frac{1}{1-\xi_l}} W^{\frac{-1}{1-\xi_l}} k_{-}{}^{\eta} \tag{4}$$

and

$$\pi(k_{-}, A|W) = (1 - \xi_l) A (A\xi_l)^{\frac{\xi_l}{1-\xi_l}} W^{\frac{-\xi_l}{1-\xi_l}} k_{-}{}^{\eta} \tag{5}$$

where $\eta = \frac{\xi_k}{1-\xi_l} < 1$. $1 - \eta$ represents the share of the profits that go to the manager's inherent ability.

Agents face uncertainty through the evolution of the firm's productivity process, A_t^i. We assume that $\log(A_t^i)$ is the sum of both a transient and persistent component

$$\log(A_t^i) = \nu_{a,t}^i + \epsilon_t^i \tag{6}$$

where

$$\nu_{a,t}^i = \rho_a \nu_{a,t-1}^i + \epsilon_{p,t}^i \tag{7}$$

Here ϵ_t^i and $\epsilon_{p,t}^i$ are the innovations to the transient and persistent component of productivity respectively. These are assumed to be i.i.d across agents.

2.2 Households

Agents are assumed to have preferences over consumption and labor given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{8}$$

with discount factor $\beta > 0$ and period utility function U. The asset market is assumed to be incomplete, with agents having access to only two assets. In addition to investing in their own firm, agents can borrow and lend from the global bond market at an interest rate R_t with natural borrowing limits.

When agents supply labor l_t^i they receive a pre-tax income $W_t l_t^i$. The agent uses post tax labor income, $(1-\tau_t^l)W_t l_t^i$, capital income, $(1-\tau_{t-1}^k)\pi(k_{t-1}^i, A_t^i|W_t)$, savings, $R_{t-1}b_{t-1}^i$, and government transfers, T_t, to fund his consumption, c_t^i, and investments, $b_t^i + k_t^i$:

$$R_{t-1}b_{t-1}^i + (1-\tau_t^k)\pi(k_{t-1}^i, A_t^i) + (1-\delta)k_{t-1}^i + (1-\tau_t^l)W_t e_t^i l_t^i + T_t = c_t^i + b_t^i + k_t^i. \tag{9}$$

The capital depreciation rate is given by δ and the capital income tax rate is τ_{t-1}^k.[4]

2.3 Government

Let B_{t-1} be the debt that the government enters with at time $t-1$. The government issues new debt, B_t, and raises labor and capital tax revenue in order to finance government transfers and payments on the previous period's debt, $R_{t-1}B_{t-1}$. This gives the budget constraint for the government as

$$B_{t-1} + (1-\tau_{t-1}^k)\int \pi(k_{t-1}^i, A_t^i|W_t)di + (1-\tau_t^l)W_t \int l_t^i di = T_t + R_{t-1}B_{t-1} \tag{10}$$

3 Equilibrium and Ramsey Plan

This section describes the conditions for a competitive equilibrium and the necessary conditions that must be satisfied by a Ramsey allocation. We begin by describing the necessary and sufficient conditions for household optimization and then the Ramsey taxation problem.

3.1 Individual Optimization

Given a sequence of prices $\{W_t, R_{t-1}\}$ and taxes $(\tau_t^l, \tau_t^k, T_t)$, sufficient conditions for optimization on behalf of the agents consist of the period by period budget constraint and the first order conditions with respect to consumption, labor, bond holdings, and capital. These imply the

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[4] Because interest rates can freely adjust the choice of the capital tax applying to the risk free bond is irrevelant.

standard optimality conditions for labor-leisure choice,

$$U^i_{c,t}(1-\tau^l_t)W_t + U^i_{l,t} = 0, \tag{11}$$

and the inter-temporal Euler equation for the risk free bond,

$$U^i_{c,t} = \beta R_t \mathbb{E}_t U^i_{c,t+1}. \tag{12}$$

In addition, optimal investment in the agent's firm implies that the following Euler equation must hold

$$U^i_{c,t} = \beta \mathbb{E}_t \left[[(1-\tau^k_t) r(k^i_t, A^i_{t+1}|W_t) + 1 - \delta] U^i_{c,t+1} \right], \tag{13}$$

where $r(k_-, A|W)$ is the marginal profits of investing an additional unit of capital in the firm given by

$$r(k_-, A|W) = \pi_k(k_-, A|W) \tag{14}$$

As the production technology of the firm has decreasing returns to scale, the marginal return to investment is less than the average return to investment with

$$\pi(k_-, A|W) = \frac{1}{\eta} k_- \pi_k(k_-, A|W) > \pi_k(k_-, A|W) \tag{15}$$

3.2 Aggregate Equilibrium Constraints

In addition to the individual optimization there are aggregate constraints that must hold for a competitive equilibrium to exist. We are studying a closed economy, so the aggregate resource constraint must hold with equality

$$\int c^i_t di + G + \int k^i_t di = \int f(k^i_{t-1}, n^i_t, A^i_t) di \tag{16}$$

All government debt must be owned by the agents, implying:

$$-B_t = \int b^i_t di.$$

This, along with feasibility, equation (16), and the household's budget constraint, equation (19a), immediately implies that the government's budget constraint, equation (10), must hold. Finally the global labor market must clear

$$\int n(k^i_{t-1}, A^i_t | W_t) di = \int l^i_t di \tag{17}$$

The supply of labor from the households must equal the labor demand from the firms. The global wage W_t will adjust so that the global labor market clears.

3.3 Allocation

At time t an agent is defined by two objects. The first is his initial bond holdings, capital holdings and permanent component of productivities: $s^i_{-1} = (b^i_{-1}, k^i_{-1}, \nu^i_a)$. The second is the sequence of shocks he has received $\vec{\epsilon}^t$. An allocation can therefore be defined as follows.

Definition 1 *An* ***Allocation*** *is a sequence of functions* $\{c_t, b_t, l_t, n_t, k_t\}$ *that map the initial state of the agent ,* s^i_{-1}*, and the sequence of productivity shocks he has received,* $\vec{\epsilon}^t$*, into the consumption, bond, labor supplied, labor hired and capital stock of the agent.*

An allocation is said to be *feasible* if equations (16) and (17) are satisfied for all times t. In both equations, *di* represents integrating over the initial distribution of states, s^i_{-1}, and the distribution of histories of individual shocks, $\vec{\epsilon}^t$.

In addition to being feasible, an allocation is said to be *individually rational* if given a sequence of government policies $\{T_t, \tau^l_t, \tau^k_t\}$ and prices (R_t, W_t) equations (19a), (11),(12) and (13) hold for all t for almost all histories of individual shocks, $\vec{\epsilon}^t$, and initial states s^i_{-1}, and the no-Ponsi conditions hold.

3.4 Competitive Equilibirium and Ramsey Problem

Given a sequence of government policies, $\{\tau^l_t, \tau^k_t, T_t\}$ we are now able to define a competitive equilibrium.

Definition 2 *Given an initial distribution of bonds, capital and permanent components of productivity,* $(b^i_{-1}, k^i_{-1}, \nu^i_{-1})$*, and a sequence of government policies,* $\{\tau^l_t, \tau^k_t, T_t\}$*, a* ***Competitive***

Equilibrium *is a sequence of prices,* $\{R_t, W_t\}$*, and a feasible allocation,* $\{c_t, b_t, l_t, n_t, k_t\}$*, such that given the prices and government policy the allocation is individually rational.*

Our goal is to solve a Ramsey taxation problem. We will assume that the Ramsey planner has a utilitarian objective function over allocations given by

$$\sum_{t=0}^{\infty} \beta^t \int \omega(s_{-1}) U\left(c_t(s_{-1}, \vec{\epsilon}^t), l_t(s_{-1}, \vec{\epsilon}^t)\right) d\Gamma_{-1}(s_{-1}) dF_t(\vec{\epsilon}^t) \tag{18}$$

where Γ_{-1} is the distribution of initial state variables, s_{-1}, and $\omega_{-1}(s_{-1})$ is the Pareto weight that the Ramsey planner places on an agent who enters the problem with initial state s_{-1}. For most of our analysis we will assume that the initial distribution of states Γ_{-1} is degenerate, and therefore the planner will have a utilitarian objective function.

Definition 3 *The* ***Ramsey Allocation*** *is the competitive equilibrium that maximizes the planners objective function* (18)*.*

3.5 The Planner's Problem

Given the definitions above it is now possible to write down the maximization problem faced by the Ramsey planner. The Ramsey planner wishes to choose an allocation $\{c_t, b_t, l_t, n_t, k_t\}$, prices $\{R_t, W_t\}$, and government policies[5] $\{\tau_t^l, \tau_t^k\}$ so as to maximize

$$\sum_{t=0}^{\infty} \beta^t \int U(c_t^i, l_t^i) di$$

subject to the constraints that the allocation be *individually rational* for each agent

$$R_{t-1} b_{t-1}^i + (1 - \tau_t^k) \pi(k_{t-1}^i, A_t^i | W_t) + (1 - \delta) k_{t-1}^i + (1 - \tau_t^l) W_t l_t^i + T_t = c_t^i + b_t^i + k_t^i. \tag{19a}$$

$$U_{c,t}^i = \beta \mathbb{E}_t \left[\Big((1 - \tau_t^k) r(k_t^i, A_{t+1}^i | W_{t+1}) + 1 - \delta \Big) U_{c,t+1}^i \right] \tag{19b}$$

$$U_{c,t}^i = \beta R_t \mathbb{E}_t U_{c,t+1}^i \tag{19c}$$

$$U_{c,t}^i (1 - \tau_t^l) W_t = -U_{l,t}^i \tag{19d}$$

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[5] Ricardian equivalence holds in this problem so that the level of Govenment debt and Transfers is indeterminate. We take the normalization that $T_t = 0$ for all $t \geq 1$.

and that the allocation must be feasible

$$\int f(k^i_{t-1}, n^i_t, A^i_t) + (1-\delta)k^i_{t-1} - c^i_t - G - k^i_t di = 0 \tag{20a}$$

$$\int l^i_t - n(k^i_{t-1}, A^i_t | W_t) di = 0 \tag{20b}$$

We have used the notation $\cdot^i$ to denote the allocation for an agent who entered with a given initial state s^i_{-1} and has received a sequence of shocks $\vec{\epsilon}^{i,t}$ at time t. Note that the individual constraints must hold for each time t for almost all initial states and histories of individual shocks. Additionally, we have assumed that $\omega^i = 1$ for all agents and that all agents enter the problem with the same level of initial capital $k^i_{-1} = k_-$ and productivity $\nu^i_{-1} = \nu_-$. This ensures that the time zero problem for the planner is trivial: setting capital and labor taxes to zero.

At this point it is worthwhile to note a few differences between this planning problem and an equivalent one with a representative agent. Ordinarily, it is worthwhile to substitute out the government policies $\{\tau^l_t, \tau^k_t\}$ and prices $\{W_t, R_t\}$ in the budget constraint of the agent, using the first order conditions of the agent. With a representative agent this would allow us to replace the sequence of individual constraints with a single implementability constraint, dropping the first order conditions of the agent and allowing the maximization to be performed solely over the allocation $\{c_t, l_t, n_t, b_t\}$. The prices and taxes could then be backed out from the allocation as those that implied optimality on behalf of the agent.

It is still possible with multiple agents to substitute out the prices and taxes in the budget constraint of the agent, but it is no longer possible to drop the first order optimality conditions of the agent. All agents face the same sequence of prices and taxes, and therefore a constraint of the planner's problem is that the associated ratios

$$\frac{-U^i_{l,t}}{U^i_{c,t}e^i_t}$$

$$\frac{U^i_{c,t}}{\beta \mathbb{E}_t U^i_{c,t+1}}$$

$$\frac{U^i_{c,t}}{\beta \mathbb{E}_t U^i_{c,t+1} r^i_{t+1}}$$

must be equalized across all agents. As there is a clear mapping from these ratios into the tax policy of the government and equilibrium prices we will keep these objects as controls of the planner's problem.

Let $x^i_{t-1} = U^i_{c,t-1}(b^i_{t-1} + k^i_{t-1})$, then equation (19a) can be rewritten as

$$\frac{x^i_{t-1}U^i_{c,t}}{\beta\mathbb{E}_{t-1}U^i_{c,t}} + U^i_{c,t}\left[(1-\tau^k_{t-1})\pi^i_t + (1-\delta)k^i_{t-1} - R_{t-1}k^i_{t-1}\right] + U^i_{l,t}l^i_t - U^i_{c,t}(c^i_t - T_t) - x^i_t = 0. \quad (21)$$

then the state variable for the planner's problem is the joint distribution over $(U^i_{c,t-1}, x^i_{t-1}, \nu^i_{a,t-1})$ and the aggregate capital stock $K_{t-1} = \int k^i_{t-1}di$. To apply our computational methods it proves convenient to decompose $U^i_{c,t-1}$ as

$$\alpha_{t-1} = m^i_{t-1}U^i_{c,t-1} \quad (22)$$

where α_{t-1} is chosen such that

$$\int \log(m^i_{t-1})di = 0$$

With this change of variables the bond pricing constraint (**??**) and the optimal firm investment (**??**) can be rewritten as

$$\alpha_t = \beta m^i_t R_t \mathbb{E}_t\left[U^i_{c,t+1}\right] \quad (23)$$

$$\alpha_t = \beta m^i_t \mathbb{E}_t\left[\left((1-\tau^k_t)r(k^i_t, A^i_{t+1}|W_{t+1}) + 1 - \delta\right)U^i_{c,t+1}\right] \quad (24)$$

The state is then split into a joint distribution over $(m^i_{t-1}, x^i_{t-1}, \nu^i_{a,t-1})$, the aggregate capital stock K_{t-1} and the aggregate marginal utility α_{t-1}.

3.6 The First Order Necessary Conditions

The planner's problem is then to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int U(c^i_t, l^i_t)di \quad (25)$$

subject to (19d) and (21-24) which ensure that the allocation is *individually rational* and equations (20) which ensure that the allocation is feasible. Let $\{\phi^i_t, \mu^i_t, \rho^i_{1,t}, \rho^i_{2,t}, \rho^i_{3,t}\}$ be the Lagrange multipliers on the individual constraints respectively and $\{\Psi_t, \xi_t\}$ be the multipliers on the aggregate constraints. The first order necessary conditions of the planner's problem with respect to the individual variables $\{c^i_t, l^i_t, m^i_t, x^i_t, k^i_{t-1}\}$ are then

$$\begin{aligned}
0 =& U^i_{c,t} + U^i_{cc,t}\Big[(1-\tau^k_{t-1}\pi^i_t + (1-\delta)k^i_{t-1} - R_{t-1}k^i_{t-1} - (c^i_t - T_t)\Big] - U^i_{c,t}\mu^i_t && \text{(26a)}\\
&+ \frac{x^i_{t-1}U^i_{cc,t}}{\beta\mathbb{E}_{t-1}U^i_{c,t}}\left(\mu^i_t - \mu^i_{t-1}\right) + m^i_t U^i_{cc,t}\rho^i_{1,t} + R_{t-1}m^i_{t-1}U^i_{cc,t}\rho^i_{2,t-1}\\
&+ m^i_{t-1}U^i_{cc,t}[(1-\tau^k_{t-1})r^i_t + 1 - \delta]\rho^i_{3,t-1} + (1-\tau^l_t)W_t U^i_{cc,t} - \xi_t\\
0 =& U^i_{l,t} - [U^i_{ll,t}l^i_t + U^i_{l,t}]\mu^i_t + U^i_{ll,t}\phi^i_t + \Psi_t && \text{(26b)}\\
0 =& \rho^i_t + \rho^i_{2,t} + \rho^i_{3,t} && \text{(26c)}\\
0 =& -\mu^i_{t-1} + \frac{\mathbb{E}_{t-1}[\mu^i_t U^i_{c,t}]}{\mathbb{E}_t U^i_{c,t}} && \text{(26d)}\\
0 =& \mathbb{E}_{t-1}\bigg\{\Big((1-\tau^k_{t-1})\pi^i_{k,t} + 1 - \delta - R_{t-1}\Big)U^i_{c,t}\mu^i_t + m^i_{t-1}U^i_{c,t}(1-\tau^k_{t-1})\pi^i_{kk,t} && \text{(26e)}\\
&- \Psi_t n^i_{k,t} + f^i_n n^i_{k,t}\xi_t + (f^i_{k,t} + 1 - \delta)\xi_t - \xi_{t-1}/\beta\bigg\}.
\end{aligned}$$

The first order conditions with respect to the aggregate variables $\{R_{t-1}, W_t, \tau^k_{t-1}, \tau^l_t, T_t\}$ are

$$\begin{aligned}
0 =& \mathbb{E}_{t-1}\int -U^i_{c,t}k^i_{t-1}\mu^i_t + m^i_{t-1}U^i_{c,t}\rho^i_{2,t-1}di && \text{(27a)}\\
0 =& \int U^i_{c,t}(1-\tau^k_{t-1})\pi^i_{W,t}\mu^i_t + m^i_{t-1}U^i_{c,t}(1-\tau^k_{t-1})\pi^i_{kW,t}\rho^i_{3,t-1} + (1-\tau^l_t)U^i_{c,t}\phi_t && \text{(27b)}\\
&- n^i_{W,t}\Psi_t + f^i_{n,t}n^i_{W,t}\xi_t di\\
0 =& \mathbb{E}_{t-1}\int -U^i_{c,t}\pi^i_t\mu^i_t - m^i_{t-1}U^i_{c,t}\pi^i_{k,t}\rho^i_{3,t-1}di. && \text{(27c)}
\end{aligned}$$

3.7 State Variables

Before we expand on how we approximate the solution to Ramsey plan, we must pay attention to the choice of state variables. Currently, we are using the joint distribution of market weights, marginal utility weighted assets and permanent component of productivities along with the aggregate capital stock and α_{t-1} as the state variables of the planner's problem. We face the immediate problem that an equilibrium may not exist for any distribution of individual state variables.

Instead, we will follow Marcet and Marimon (2011) and replace the individual state variable x^i_{t-1} with it's co-state variable: the associated multiplier on the budget constraint of the agent, μ^i_{t-1}. For the remainder of the paper we will take the state variable of the planner's problem to be the joint distribution Γ_{t-1} over $(m^i_{t-1}, \mu^i_{t-1}, \nu^i_{a,t-1})$, the aggregate capital stock K_{t-1}, and the aggregate level of marginal utility α_{t-1}. This choice of state variables proves particularly advantageous as, in the no-shock limit, the law of motion for both the market weights m^i_t and the multipliers μ^i_t satisfy

$$m^i_t = m^i_{t-1} \text{ and } \mu^i_t = \mu^i_{t-1}.$$

In the no-shock limit we have a complete markets economy, and therefore the ratio of marginal utilities, and hence market weights, will be constant over time. μ^i_t is constant as, in the absence of individual shocks, the sequence of budget constraints for the agent can be written as a single time zero budget with a constant multiplier.

4 Perturbation Theory with Heterogeneous Agents

A significant contribution of this paper is the use of perturbation theory to solve a Ramsey taxation model with both idiosyncratic and aggregate risk. This is an application of the theory outlined in Evans (2014), which goes into the specific details of how to implement the approximation. This section will be an overview of the approximation method with specific detail paid to how the assumptions of Evans (2014) are satisfied.

4.1 General Setup

Our goal with this algorithm is to solve models characterized by the following system of equations

$$F(z_{t-1}, y_t, e_{t-1}, v_t, \epsilon_t, Z_{t-1}, Y_t, \Theta_t | q) = 0 \tag{28}$$

and

$$\int G(z_{t-1}, y_t, \epsilon_t, Z_{t-1}, Y_t, \Theta_t|q)dz_{t-1}, d\epsilon_t = 0 \tag{29}$$

where F describes a sequence of constraints for each individual agent and G represents a set of aggregate constraints for the economy. In the notation we are using, individual variables are represented by lowercase letters. z_{t-1} are the state variables for the individual's problem, y_t are the controls at time t, e_{t-1} is the $t-1$ expectation of some function of the controls

$$e_{t-1} = \mathbb{E}_{t-1}f(y_t), \tag{30}$$

v_t is a subset of the controls y_{t+1} that are assumed to be measurable with respect to time t (which can be imposed using e_{t-1}), and finally, ϵ_t are the idiosyncratic shocks. Aggregate variables are denoted by capital letters with Z_{t-1} representing the aggregate state variables (such as capital stock), Y_t are the aggregate control variables and Θ_t are the aggregate shocks. Furthermore we have a perturbation parameter, q, that controls the size of the idiosyncratic and aggregate shocks. It is assumed that the variance of these shocks approaches zero as $q \to 0$.

Next period's state vectors are a subset of the controls, and hence, without loss of generality we assume that the first n_z elements of y_t are next period's individual state z_t. Similarly, the first n_Z elements of Y_t are assumed to be next period's aggregate state Z_t. We define the matrices I_y^z and I_Y^Z as the projections such that $z_t = I_y^z y_t$ and $Z_t = I_Y^Z Y_t$. Finally, the previous period's forward looking variable v_{t-1} is also a subset of the controls y_t. We assume that the last n_v elements of y_t are the previous period's forward-looking variables v_t, and define the matrix I_y^v to be the projection such that $v_t = I_y^v y_{t+1}$.

Note that in this environment the equations describing the economy are split into two types: individual constraints that have to hold for almost all agents and aggregate constraints on the entire economy. The equations governing our current setting can be divided into these two groups. First the individual constraints of the planner's problem equations (19d) and (21-24) and the first order conditions with respect to the individual controls (26) give the equations that make up F in equation (28). Next the aggregate constraints of the planner's problem (20) and the first order conditions with respect to the aggregate variables (27) make up G in equation

(29). The expectation terms such as $\mathbb{E}_{t-1}U^i_{c,t}$, $\mathbb{E}_{t-1}U^i_{c,t}\mu^i_t$ and $\mathbb{E}_{t-1}U^i_{c,t}r^i_t$ can be computed using equation (30).

Equation (28) makes a significant structural assumption that will make the problem computationally tractable. Specifically, that the constraints pertaining to one individual agent only depend on other agents through their effect on the aggregate variables. This symmetry can be observed in equations (21-19d) and (**??**-19d). Specifically the constraints and first order conditions of an agent of state $(m^i_{t-1}, \mu^i_{t-1}, \nu^i_{a,t-1})$ only depend on his control variables, his state, the individual shocks, and the aggregate control variables.

We are looking to approximate the individual policy rules

$$y_t = y\left(\epsilon_t, \Theta_t \middle| z_{t-1}, \Gamma_{t-1}, Z_{t-1}, q\right) \tag{31}$$

and the aggregate policy rules

$$Y_t = Y\left(\Theta_t \middle| z_{t-1}, \Gamma_{t-1}, Z_{t-1}, q\right). \tag{32}$$

that solve the system of equations (28) and (29) using perturbation theory. Perturbation theory has, in the past, been applied to economies with uninsurable income risk: Preston and Roca (2007); Mertens and Judd (2013); Kim et al. (2010). Our approach will differ in two key ways. The first is the point around which the economy is approximated; rather than approximating around a degenerate distribution $\overline{\Gamma}$ we will use the closest distribution to Γ_{t-1} with a non-stochastic steady state and approximate policy rules around that distribution. This will prove necessary to accurately approximate the Ramsey allocation.

Our algorithm also differs in how the derivatives of the optimal policy rules are computed. The main difficulty in applying perturbation theory to economies with heterogeneous agents is that it is necessary to know how the allocation y^i for agent i changes with respect to a change in the state of agent j

$$\frac{\partial y^i}{\partial z^j}.$$

Without some simplification there would be far to many derivatives to compute, and the problem would not be computationally tractable. In other papers this is solved by approximating the policy rules around a degenerate distribution. Under this assumption

$$\frac{\partial y^i}{\partial z^j} = \frac{\partial y^i}{\partial z^k}$$

for all $j \neq k$. Unfortunately this approach cannot be applied when approximating around a non-degenerate distribution and therefore cannot be applied here. Instead we develop a weaker assumption which allows us to decompose derivatives such as $\frac{\partial y^i}{\partial z^j}$ into two computationally tractable components

$$\frac{\partial y^i}{\partial Y}\frac{\partial Y}{\partial z^j}.$$

This has the added benefit of being applicable to a wider range of economic problems, including heterogeneous agent problems where agents have permanent types, differings in both preferences and productivities. Evans (2014) provides an example where this methodology can be applied to a multi-country model with both complete and incomplete markets.

4.2 Point of Approximation

When using perturbation theory to approximate an economic model with forward looking agents, the approximation needs to be performed around a steady state. The goal is to take the known solution of a simpler model and then use Taylor series expansions, around that simpler model, to approximate the more complicated model of interest. For this analysis we will be approximating the Ramsey planner's problem in the joint limit as $\sigma_\epsilon \to 0$ and $\rho_a \to 1$.[6] This problem has the very convenient interpretation as the continuation of a Ramsey planner's problem with permanent productivity types. We wish to study this limit for two reasons. First, it will provide us with a steady state around which we can apply our perturbation analysis. Second, it will provide us with a simpler problem which we can use to understand the economic forces governing the planner's problem.

The no shock limit allows us to pin down the evolution of the Pareto weights m_t^i and the

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[6] For this reason we will be exploring calibrations where the permanent component of the productivity process is close to 1. We are working on extensions where this assumption can be relaxed.

multiplier on the household's budget constraint μ_t^i. With the first order condition with respect to x_{t-1}^i we derived that μ_t^i follows a twisted Martingale

$$\mu_{t-1}^i = \frac{\mathbb{E}_{t-1} U_{c,t}^i \mu_t^i}{\mathbb{E}_{t-1} U_{c,t}^i}.$$

In the limit as the size of the idiosyncratic shocks approaches zero, the marginal utility terms in the numerator and denominator will cancel, leaving us with

$$\mu_{t-1}^i = \mu_t^i.$$

Therefore, in the permanent type limit the multiplier on the agent's budget constraint is a constant μ^i. Similarly, in the limit as the size of the idiosyncratic shocks approaches zero, we can combine the definition of Pareto weights $\alpha_t = m_t^i U_{c,t}^i$ and the bond pricing constraint $\alpha_{t-1} = \beta R_{t-1} m_{t-1}^i U_{c,t}^i$ to obtain

$$m_{t-1}^i = \frac{\alpha_{t-1}}{\beta R_{t-1} \alpha_t} m_t^i.$$

The normalization that $\int \log(m_t^i) di = 0$ then immediately implies

$$\frac{\alpha_{t-1}}{\beta R_{t-1} \alpha_t} = 1 \tag{33}$$

or

$$m_{t-1}^i = m_t^i.$$

Thus the Pareto weight for agent i, m^i, is constant for all time. Both of these results are intuitive. In the limit as the size of the idiosyncratic shocks approaches zero we have a complete markets economy and thus the ratio of marginal utilities will be constant for all time. This immediately implies that the Pareto weights will be constant. Similarly, under complete markets we can write the sequence of budget constraints as a single time zero budget constraint the multiplier on which corresponds to μ^i.

In addition, under the joint limit $\sigma_\epsilon \to 0$ and $\rho_a \to 1$, the evolution of the persistent component of productivities follows

$$\nu^i_{a,t} = \nu^i_{a,t-1}$$

The associated productivities of the agents, $A^i = \exp(\nu^i_a)$, are then also permanent in the non-stochastic limit. Solving the Ramsey planner's problem in the non-stochastic limit is then equivalent to solving for the non-stochastic steady state of the continuation of a Ramsey planner's problem where the agent's enter with a distribution of permanent productivities A^i, market weights m^i and multipliers on the time zero budget constraint μ^i.

Steady State

In solving for a non-stochastic steady state we are solving the continuation of a Ramsey taxation problem with permanent types for a given joint distribution, $\overline{\Gamma}$, of Pareto weights (m^i), multipliers (μ^i), and productivities (A^i). The state of this problem is the level of aggregate capital K_{t-1} and aggregate marginal utility α_{t-1} and we would therefore look for a mapping $K_{t-1}, \alpha_{t-1} \mapsto (K_t, \alpha_t, R_{t-1}, \tau^k_t, \tau^l_t, W_t, \xi_{t-1}, \eta_t)$ that solves the non-stochastic system of equations.[7]

While the global solution can be found via projection methods, we are interested in computing the steady state of the planners problem. This amounts to solving for the root of a non-linear function of the aggregate variables $(\overline{K}, \overline{\alpha}, \overline{\tau}^k, \overline{\tau}^l, \overline{W}, \overline{\xi}, \overline{\Psi})$.[8] Given a guess for $(\overline{K}, \overline{\alpha}, \overline{\tau}^k, \overline{\tau}^l, \overline{W}, \overline{\xi}, \overline{\Psi})$ it is possible to solve for the policies $(\overline{c}^i, \overline{l}^i, \overline{m}^i, \overline{x}^i, \overline{k}^i)$ that are individually rational for each $(\overline{m}^i, \overline{\mu}^i, \overline{A}^i)$ in the support of $\overline{\Gamma}$ as well as the individual multipliers $(\overline{\phi}^i, \overline{\mu}^i, \overline{\rho}^i_1, \overline{\rho}^i_2, \overline{\rho}^i_3)$. We then solve numerically for the $(\overline{K}, \overline{\alpha}, \overline{\tau}^k, \overline{\tau}^l, \overline{W}, \overline{\xi}, \overline{\Psi})$ that solve the non-stochastic aggregate constraints.

While analytically we cannot prove the existence of a steady state, the existence and uniqueness of a steady state is numerically robust for any distribution $\overline{\Gamma}$. Note that different distributions $\overline{\Gamma}$ will have different steady state levels of capital $\overline{K}$, which will allow us to more accurately track the evolution of capital over time.

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[7] Note given the aggregate variables $(K_t, \alpha_{t-1}, R_{t-1}, \tau^k_t, \tau^l_t, W_t, \xi_{t-1}, \psi_t)$ one can back out the individual policy rules at each time t and, hence, also compute the individual Lagrange multipliers.

[8] Equation (33) implies that in the steady state $\overline{R} = \frac{1}{\beta}$.

Marginal Benefits of Redistributionary Taxation

We've seen above that the distribution of multipliers μ^i on the marginal utility weighted budget constraint jointly determines the steady policy of the government. These multipliers enter into the first order condition with respect to capital taxes

$$-\int \beta \pi_t^i \mu_t^i U_{c,t}^i + \rho_{3,t-1}^i m_{t-1}^i U_{c,t}^i r_t^i di = 0$$

The first term is comprised of the $-\mu^i U_{c,t}^i$ times the tax base of individual i. $-\mu^i U_{c,t}^i$ therefore represents the marginal benefit, to the planner, of extracting resources from agent i at time t. The multipliers, $-\mu^i$, therefore capture the marginal redistributionary benefits of distortionary taxation.[9]

4.3 Symmetry

The main difficulty faced in applying perturbation theory to solve models with large numbers of agents is the number a derivatives that need to be computed. In approximating the policy rules via a Taylor expansion around a steady state we need to compute derivatives, of first and higher order, of the form $\frac{\partial y^i}{\partial z^j}$. It would not be computationally tractable to compute every derivative of this type.[10] As noted above, this problem has been solved in the past by approximating around a degenerate distribution, which makes most of these derivatives identical. We've found that approximating around a degenerate distribution does not provide an accurate approximation of the Ramsey plan, and therefore a new approach is required.

Following Evans (2014), we decompose the derivatives $\frac{\partial y^i}{\partial z^j}$ into two simpler and far easier to compute components[11]

$$\frac{\partial y^i}{\partial Y}\frac{\partial Y}{\partial z^j}.$$

This decomposition requires a far less restrictive assumption

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[9] By the marginal redistributionary benefits of distortionary taxation we refer specifically to the benefits of extracting resources from the budget constraint of agent i. We will see later that a utilitarian planner will have other motives for instituting distortionary labor taxes.

[10] For example, if the distribution of agents were approximated by only 100 agents there would be 10000 derivatives that needed to be computed for the first order expansion alone.

[11] This is written heuristically, for the precise decomposition see Evans (2014)

Assumption 1 *The partial derivative*

$$I_y^z \frac{\partial y}{\partial z}\left(0,0|\overline{z}^i,\overline{\Gamma},\overline{Z},0\right)$$

is independent of $\overline{z}^i$ *for almost all* $\overline{z}^i$ *in the support of* $\overline{\Gamma}$.

In other words, in the non-stochastic limit, the derivative of the state tomorrow with respect to the state today has to be independent of the current state of that agent. In our particular setup, we have shown that, in the non-stochastic limit, each state variable is constant over time. Therefore, the derivative in assumption 1 will be the four dimensional identity matrix, independent of the of the state $\overline{z}^i$ of the agent.

In addition to Assumption 1, two additional assumptions can be made to reduce the computational difficulty of the problem. Specifically:

Assumption 2 *The policy rules for the individual state variables in the non-stochastic limit,* $q \to 0$,

$$z\left(0,0|\overline{z}^i,\overline{\Gamma},\overline{Z},0\right)$$

is independent of Z_{t-1}

and

Assumption 3 *The policy rules*

$$z_t = z\left(0,0|\overline{z}^i,\overline{\Gamma},\overline{Z},0\right)$$

are independent of Γ_{t-1} *when* $q = 0$.

It is easily verified that both of these assumptions are satisfied as well.

5 Persistent Investment Risk

We are now able to explore the role that the persistence of investment risk plays in the optimal allocation chosen by the Ramsey planner. Other papers have studied the role of capital and income taxation in the presence of uninsurable investment risk, such as Panousi and Reis (2014) and Itskhoki and Moll (2014) but it has been in the presence of i.i.d. productivity shocks. Making productivity shocks i.i.d. is very helpful in making the planner's problem tractable, as the returns to investment will be decoupled from the distribution of assets, which makes for a convenient aggregation.[12] We shall see, however, that this assumption significantly affects how the planner uses capital taxes, as it reduces the value of using capital taxes to redistribute resources from agents who have received a series of positive productivity shocks to those who have received a series of negative productivity shocks.

5.1 Capital Taxes when Investment Risk is i.i.d.

This paper is closest in spirit to Panousi and Reis (2014) in which they study the long run Ramsey allocation in a problem with i.i.d. investment risk. Aside from the persistence of investment risk our paper differs in several substantial ways. First, while this paper is in discrete time they study the optimal Ramsey allocation in continuous time. The households in their paper supply labor supply inelastically, and aggregate labor supply is normalized to 1. Finally households operate firms with a constant returns to scale technology and i.i.d shocks to the capital depreciation rate.

As a point of comparison, we adjust the model laid out in section 3 to match, as close as possible, their environment. We begin by making the households supply a unit of labor inelastically, and setting their preferences to be log utility over consumption. To approximate continuous time we choose a short the period length by setting the discount factor β at 0.99. Finally, we change the TFP shocks the agents face into shocks to the capital depreciation rate. Our approximation method does not allow us to study constant returns to scale productions, but we choose a production function close to constant returns with $\xi_l + \xi_k = 0.8$.[13]

While we would not expect to replicate the results of Panousi and Reis exactly, there are specific qualitative features of Ramsey plan that we would expect to capture. First, they find

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[12] Specifically, along with the assumption of log utility preferences for the entrepreneurs and constant returns to scale technology of the production function.

[13] We will discuss in later sections why decreasing returns to scale production is necessary. Either way, changing the scale does not seem to have an appreciable qualitative effect on the results.

that in the long run the government will choose to subsidize capital investment, with the size of the subsidy increasing in the variance of the depreciations shocks.[14] They also find that the optimal tax rate will be initially larger and then decrease to its long run level over time.

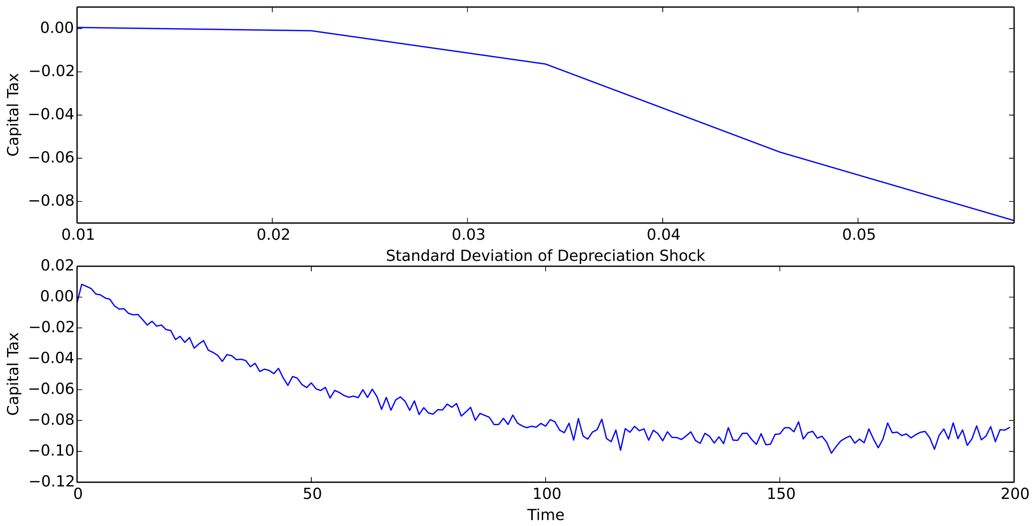

Figure 1: Tax rate after 200 periods (above) as a function of the standard deviation of the capital depreciation shock, σ_ϵ, and the transition path (below) of the capital tax rate starting from all agents being identical.

To test this we simulate our economy from the initial distribution where all agents are identical for 200 periods for various levels of the standard deviation of the depreciation shock, σ_ϵ. Figure 1 plots both the capital tax rate after 200 periods as a function of σ_ϵ, and the transition dynamics of the capital tax rate when $\sigma_\epsilon = 0.058$. We see that, as σ_ϵ increases, the government increases the capital subsidy in the long run. Moreover, we can confirm the transition dynamics, as the capital tax rate initially is higher and then decreases along the transition path.

The driving force behind the capital tax rate in Panousi and Reis lies in how investment risk affects the savings decisions of the agent. As in Angeletos (2007), the presence of investment risk can lead agents to under-invest relative to first best. In a similar mechanism to Aiyagari (1995) the government will move to correct these savings decisions using the capital tax rate. If the

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[14] This is only true in their paper if the variance of the depreciation shock is small enough. Once the variance of the shock goes beyond a certain threshold the long run capital tax is increasing in the variance of the depreciation shock.

variance of the capital depreciation shock is not too large then the government can encourage investment through capital subsidies.[15]

One key difference between the discrete time and continuous time environments is that in discrete time the shock lasts for the length of the period, reducing the ability of the agents to smooth consumption using debt. This can be interpreted as the persistence of the shock. As β decreases the effective persistence of the shock increases, until we reach the extreme limit when $\beta = 0$ of the shock lasting the entire lifetime of the agent. Figure 2 plots the capital tax rate after 200 periods as a function of the model's discount factor holding constant σ_ϵ at 0.058. We see that the long run capital tax rate is decreasing in the discount factor β. As we move away from $\beta \approx 1$, the size of the capital subsidy decreases, with the planner eventually choosing to set a positive tax rate on capital income. If the effective persistence of the shock process is large enough the planner will choose to tax capital rather than subsidize it. We shall see later that this is a result of the planner using capital taxes to redistribute the investment shocks across agents. When the period length is short, that motive is small relative to the planner's desire to correct the under-investment on behalf of the agents, but as the period length increases, and

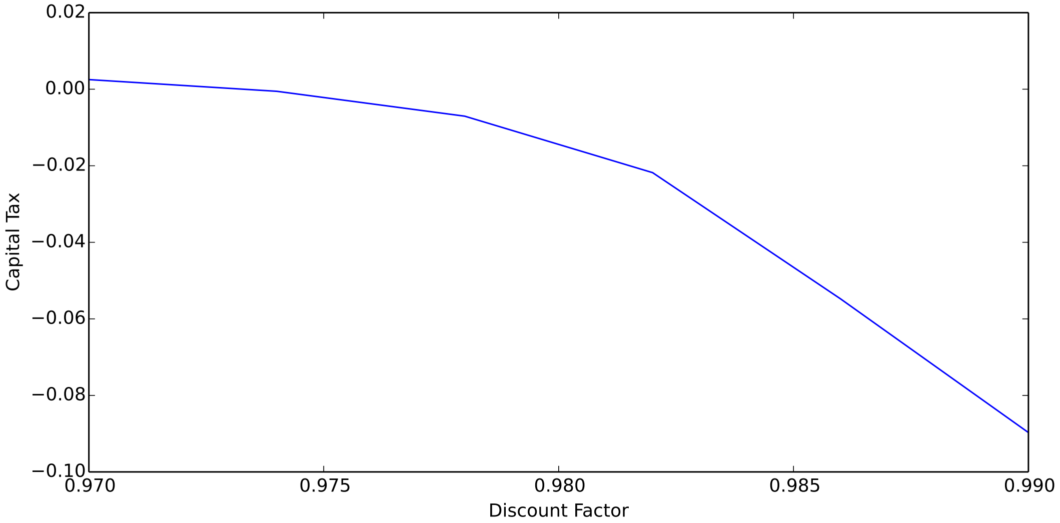

Figure 2: Capital tax rate after 200 periods as a function of the agent's discount factor β.

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[15] If the depreciation shocks are large enough then increasing the capital tax can encourage investment by decreasing the return on the risk-free interest rate. This requires very large depreciation shocks and is beyond the scope of our perturbation methods, so will not be explored in this paper. In any case, we shall soon see that the need to redistribute persistent shocks will dominate these general equilibrium effects.

hence the effective persistence, eventually the insurance motive dominates.

Constant vs Decreasing Returns to Scale

Panousi and Reis follow Angeletos 2007 in choosing the model entrepreneurs with a constant returns to scale production function, while in this paper we choose to model entrepreneurs with a decreasing returns to scale production function. Decreasing returns to scale is not an uncommon assumption in the financial friction literature. We can motivate this assumption by assuming that managerial ability is a component of the production function that cannot be replicated, making it impossible for the entrepreneurs to scale up production indefinitely.

While constant returns to scale can provide some significant analytical tractability as in Moll (2014); Itskhoki and Moll (2014); Panousi and Reis (2014), in this scenario the decreasing returns to scale production function provides us with computational tractability when performing a perturbation analysis. Specifically, the problem is that with constant returns to scale the portfolio decision for an individual agent between investing in her private firm and purchasing a bond is indeterminate. That means that in the no-shock limit there will be multiple non-stochastic steady state policy rules corresponding to the investment decisions of each agent. Most importantly, unless the correct portfolio is chosen the non-stochastic limit may be discontinuous which would mean that applying the implicit function theorem to compute derivatives would be invalid.

Additionally, decreasing returns to scale will prove important when approximating the Ramsey allocation with persistent productivity shocks. In solving that model we approximate policy rules around the limit where there are permanent differences in firm productivity across agents. It is only with a decreasing returns to scale production function that such a non-stochastic limit exists.

5.2 Capital Income Taxes with Persistent Investment Risk

In the i.i.d. case we've seen how decreasing the discount factor and, hence, increasing the effective persistence of the productivity shocks, increases the long run capital tax rate chosen by the Ramsey planner. This motivates studying the role that persistence of the productivity process plays in the optimal tax plan chosen by the Ramsey planner. In doing so we will return to the economy spelled out in Section 2 of this paper.

We wish to determine how the long run capital income tax varies with the persistence of the productivity process faced by the entrepreneurs. Recall that a firm's productivity follows

$$\log(A_t^i) = \nu_{a,t}^i + \epsilon_t^i$$

where

$$\nu_{a,t}^i = \rho_a \nu_{a,t-1}^i + \epsilon_{p,t}^i.$$

By choosing $\rho_a = 0.99$, the process for $\log(A_t^i)$ is a mixture of a highly persistent and highly transitory component. We can then change σ_ϵ^2 and $\sigma_{\epsilon_p}^2$, the variances of ϵ_t^i and $\epsilon_{p,t}^i$ respectively, to vary the overall persistence of $\log(A_t^i)$ while holding constant the cross-sectional variance of productivities.

We choose the period utility of the agent to be

$$U(c,l) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$$

As such we have 7 parameters to calibrate: the discount factor β, the risk aversion parameter σ, the disutility of labor γ, the depreciation rate δ, the production parameters ξ_l and ξ_k and the cross-sectional variance of log productivities σ_a^2. We set the discount factor at 0.95 and the coefficient of relative risk aversion, σ, at 1.5. We choose $\gamma = 2$ in order to get a Frisch elasticity of 0.5. The most non-standard parameters of this model are the two production parameters ξ_l and ξ_k as well as the cross-sectional variance of log productivity. We base these choices after the Buera and Shin (2013) who calibrate a similar model to match the wealth distribution of the U.S. without financial frictions. We match the same moments, and therefore choose $\xi_l = 0.66 \cdot 0.81$ and $\xi_k = 0.34 \cdot 0.81$, where 0.81 represents the share of productivity going to the labor and capital inputs and 0.66 and 0.34 represents the share of that amount going to labor and capital respectively. In future sections we explore how the choice of the span of control of 0.79 affects the tax plan chosen by the Ramsey planner. Finally, we choose $\sigma_a^2 = 0.11$ to match the variance of firm size, as in Buera and Shin (2013). While there is not consensus on the on the autocorrelation on idiosyncratic productivity process the values common to the literature are generally in the rage of $[0.7, 0.9]$.[16] For the remainder of the paper we will report results for a range of autocorrelations, but the focus should be on that range.

Long Run Capital Income Tax

In Figure 3 we simulate the economy for 300 periods from the initial condition where all agents are identical for varying levels of the autocorrelation of the productivity process. It is clear from these dynamics that when the autocorrelation is low the capital income tax rate starts out and remains low throughout the course of the simulation. However, for empirically relevant levels of autocorrelation, the capital income tax rate becomes large after 300 periods, on the order of 35-40%. We can capture the driving force behind this by observing the first order condition with respect to the capital income tax in equation (**??**).

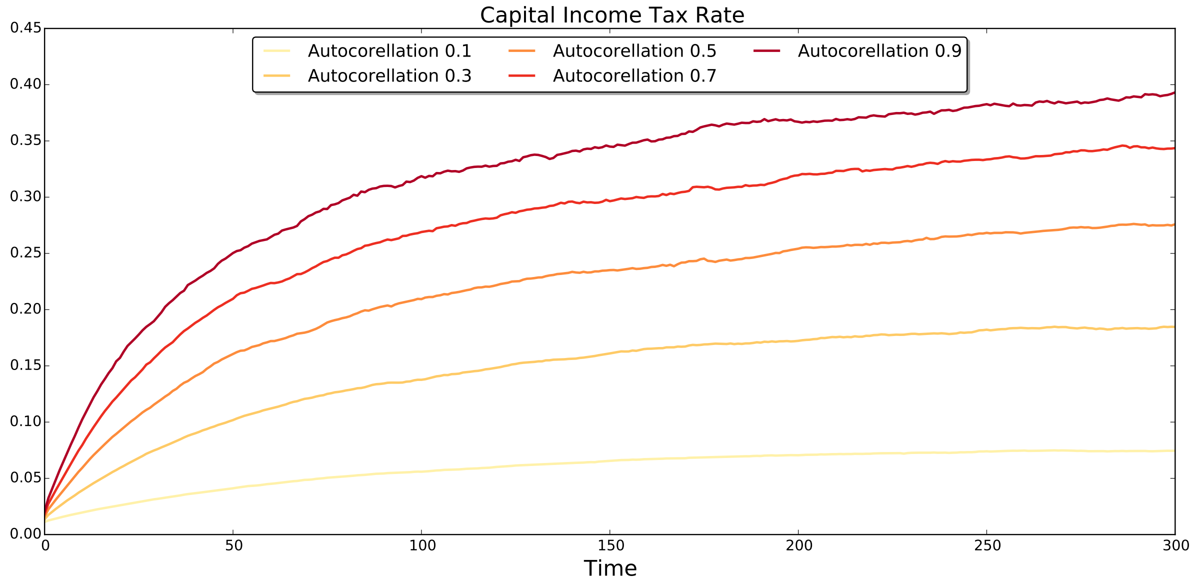

Figure 3: The capital income tax after 200 periods as a function of the persistence of the productivity process.

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[16] see Moll (2014)

The first order condition with respect to the capital income tax is

$$-\int \beta\pi_t^i\mu_t^iU_{c,t}^i+\rho_{3,t-1}^im_{t-1}^iU_{c,t}^ir_t^idi=0$$

The first part of this formula

$$-\int \beta\pi_t^i\mu_t^iU_{c,t}^idi$$

represents the marginal benefit of direct redistribution through taxation, while

$$-\int \rho_{3,t-1}^im_{t-1}^iU_{c,t}^ir_t^idi$$

captures the average marginal cost of distorting the capital accumulation equation of agent i.. $-\mu_t^iU_{c,t}^i$ is the marginal value to the time 0 planner of extracting resources from agent i at time t. π_t^i is the current gross capital income of agent i. This constitutes the government's tax base with respect to the capital income tax for agent i. The existence of transfers implies that the average value of $\mu_t^iU_{c,t}^i$ will be zero for all time

$$\int \mu_t^iU_{c,t}^idi=0.$$

so the redistributive motive is the covariance of need for redistribution, $U_{c,t}^i\mu_t^i$, with the tax base π_t^i. Recall that the μ_t^i follows a twisted Martingale

$$\mu_{t-1}^i=\mathbb{E}_{t-1}\left[\frac{U_{c,t}^i}{\mathbb{E}_{t-1}U_{c,t}^i}\mu_t^i\right]$$

Thus μ_t^i inherits costs of previous commitments to redistribution taxation from μ_{t-1}^i. This link to the past is key to understanding the difference in magnitude between the i.i.d. case and the near unit root case. Decompose the redistributional motive for the capital income tax into

$$-\int \beta\pi_t^i\mu_t^iU_{c,t}^idi=-\int \beta\pi_t^i(\mu_t^iU_{c,t}^i-\mu_{t-1}^iU_{c,t-1}^i)di-\int \beta\pi_t^i\mu_{t-1}^iU_{c,t-1}^idi.$$

Consider an agent who has received a sequence of good productivity shocks in the past and therefore the government would like to extract resources through distortionary taxation, i.e., for him $\mu^i_{t-1} < 0$. If the productivity process is i.i.d. then A^i_t is uncorrelated with $\mu^i_{t-1}U^i_{c,t-1}$ and thus

$$\int \beta\pi^i_t\mu^i_{t-1}U^i_{c,t-1}di \approx 0^{17}$$

When productivity is i.i.d. tax policy today only allows the planner directly to redistribute income directly from agents who were lucky today. With persistent productivity shocks productivity shocks we recover a correlation between π^i_t and $\mu^i_{t-1}U^i_{c,t-1}$. An agent who has received good productivity shocks in the past will have a higher A^i_t and also promises of redistributional taxation from the planner, $\mu^i_{t-1}U^i_{c,t-1} < 0$ implying

$$\int \beta\pi^i_t\mu^i_{t-1}U^i_{c,t-1}di < 0$$

The planner will therefore have an incentive to implement a higher capital income tax, since it will allow her not only to redistribute from agents who were lucky today, but also from agents who received good productivity shocks in the past. In addition, Appendix A explores the response of the tax rate with the introduction of financial friction, and finds that the decision to have large capital taxes with high persistence is robust to the inclusion of financial frictions.

Capital Tax Decomposition

In choosing the optimal capital tax the planner is balancing the benefits of redistribution with the possible efficiency costs of the higher tax rate. The following proposition allows us to decompose the tax wedge into two terms exactly capturing these two conflicting motives. Define $r^{e,i}_t = (1-\tau^k_{t-1})r^i_t + 1 - \delta - R_{t-1}$ as the excess marginal returns to capital, then

Proposition 1 *The capital tax wedge can be written as*

$$\frac{\tau^k_{t-1}}{1-\tau^k_{t-1}} = \hat{\tau}^{R,k}_{t-1} + \hat{\tau}^{E,k}_{t-1}$$

---

[17] It will not be exactly zero as the savings decisions of the agent, k^i_t, may depend on the history of previous shocks which will induce a correlation between π^i_t and $\mu^i_{t-1}U^i_{c,t-1}$, but this will be dominated by the other term.

where

$$\hat{\tau}^{R,k}_{t-1} = -\frac{\int (k^i_{t-1} r^{e,i}_t + (1-\eta)(1-\tau^k_{t-1})\pi^i_t)\mu^i_t U^i_{c,t} di}{K_{t-1}(R_{t-1} - 1 + \delta)} \tag{34}$$

and

$$\hat{\tau}^{E,k}_{t-1} = -\frac{R_{t-1}\xi_t - \frac{\xi_{t-1}}{\beta} - \int k^i_{t-1} \frac{cov(U^i_{c,t}, f^i_{k,t})}{\mathbb{E}_{t-1} U^i_{c,t}} di}{K_{t-1}(R_{t-1} - 1 + \delta)}. \tag{35}$$

The proof is relegated to the appendix. $\hat{\tau}^{R,k}_{t-1}$, the contribution to the capital tax wedge of the restributional motive, is proportional covariance of $k^i_{t-1} r^{e,i}_t + (1-\eta)(1-\tau^k_{t-1})\pi^i_t$ with $\mu^i_t U^i_{c,t}$. $k^i_{t-1} r^{e,i}_t + (1-\eta)(1-\tau^k_{t-1})\pi^i_t$ combines the excess returns to capital with the proportion of profits allocated to the individual managers inherent ability. This term, therefore, captures the covariance of the marginal value of of extracting resources from agent i with the excess returns to capital for agent i beyond the risk-free rate.

In addition to the redistributional term, $\hat{\tau}^{E,k}_{t-1}$ captures the efficiency cost of the capital income tax. The first part, proportional to $R_{t-1}\xi_t - \frac{\xi_{t-1}}{\beta}$, captures the positive long run capital tax in Aiyagari (1995). As agents face idiosyncratic income risk, $R_{t-1} < \beta^{-1}$ in the long run which pushes $\tau^{k,E}_{t-1}$ towards a positive capital income tax. With investment risk, entrepreneurs bear the risk of additional investment which can lead to under-saving. The last component, proportional to

$$\int k^i_{t-1} \frac{\text{cov}(U^i_{c,t}, f^i_{k,t})}{\mathbb{E}_{t-1} U^i_{c,t}} \xi_t di$$

captures the affect of under saving in models with idiosyncratic investment risk. In order to correct the under saving, the planner has an incentive to subsidize capital.

Figure 4 plots two three components as a function of the autocorrelation of the firm's productivity process. When the productivity process is close to i.i.d. the efficiency and the insurance terms are close to balance, favoring a slight capital tax. As the autocorrelation increases, however, both terms increase and the planner moves towards large capital income taxes.[18] The lion's share of the change in the capital income tax is driven by the planner's motive for ex-post redistribution with a small component coming from inter-temporal efficiency.

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[18] Recall that the under-saving results of Angeletos (2007) were only true for a subset of the parameter space and were only explored in the presense of i.i.d. investment risk. To my knowledge no one has explored how the inclusion of persistent productiivty shocks effects this.

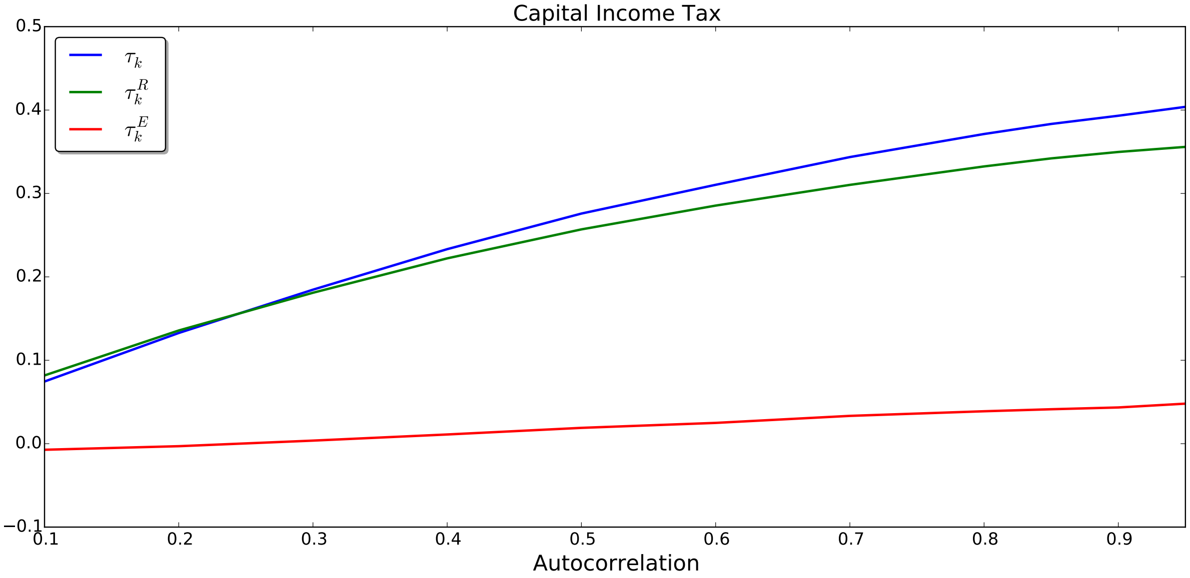

Figure 4: Decomposition of the capital income tax after 300 periods, τ^k into its two components: $\tau^{k,R}$ and $\tau^{k,E}$

Labor Tax

In our model the Ramsey planner has two taxation instruments to help increase the ex-ante utility of the agents. We've already seen how the long run tax rate increases in the persistence of the productivity process that the agents face. An open question is how the planner should use distortionary labor taxes. At first glance, one would expect the Ramsey planner to subsidize labor income. Our simulations use CES preferences over consumption and labor

$$U(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$$

These preferences have a wealth effect. An agent whose firm has been particularly productive will have high consumption, and hence, will work less than an agent whose firm has been unproductive. A labor subsidy plus a lump sum tax would allow the government directly to redistribute resources from agents with high consumption to agents with low consumption. In Figure 5 we see the transition dynamics of the labor income tax over 300 periods for varying levels of the autocorrelation of the firm productivity process. For low levels of autocorrelation our intuition does hold true; the government subsidizes labor income. As the persistence of the

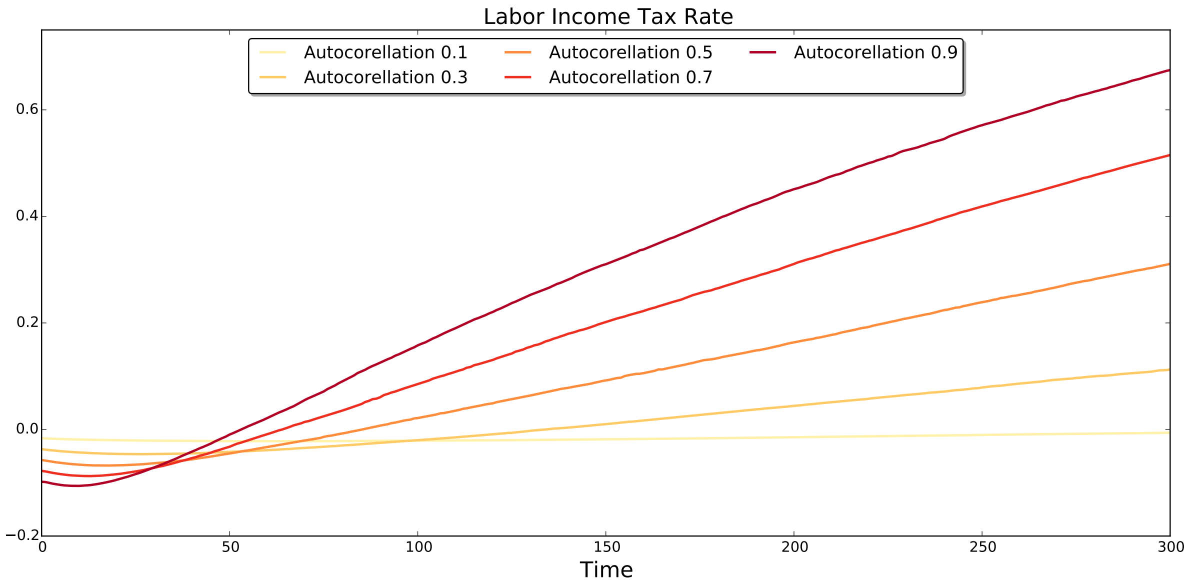

Figure 5: The transition dynamics of the labor income tax over 300 periods.

productivity process increases the government begins to implement a large labor tax.

The following proposition allows us to decompose the labor income tax rate into two terms, which will allow us to understand why the planner chooses to set such a large tax rate on labor income despite there being now heterogeneity in labor productivity.

Proposition 2 *The labor income tax rate can be decomposed as*

$$\tau_t^l = \tau_t^{R,l} + \tau_t^{E,l}$$

where

$$\tau_t^{R,l} = -\frac{\int (1+\gamma) W_t l^i t U_{c,t}^i \mu_t^i di}{\int W_t U_{c,t}^i l_t^i di}$$

and

$$\tau_t^{E,l} = \frac{\int l_t^i W_t (U_{c,t}^i - \xi_t) di}{\int W_t U_{c,t}^i l_t^i di}$$

The first term $\tau_t^{R,l}$ is proportional to the covariance of the distribution of labor earnings across agents with the marginal benefit of extracting resources from them through taxation. This term captures our intuition, that if the poorest agents are working hard the planner will want to subsidize labor income.

But the planner also need to take into account general equilibrium effects. The second term concerns the equitable distribution of resources and compares the value if labor output were distributed directly to the agents who produce it ($\int l^i_t W_t U^i_{c,t} di$) vs it's value in relaxing the resource constraint ($\int l^i_t W_t \xi_t$). If the latter is smaller the planner will wish to tax labor income.

To see how these terms contribute to the labor tax we plot the decomposition of the labor tax rate as a function of the autocorrelation of the firm's productivity process in Figure 6. Both $\tau^{l,I}$ and $\tau^{l,E}$ contribute positively to the tax rate chosen by the Ramsey planner while, as our initial intuition suggested, the redistributive component, $\tau^{l,R}$, pushes the planner towards a labor subsidy.

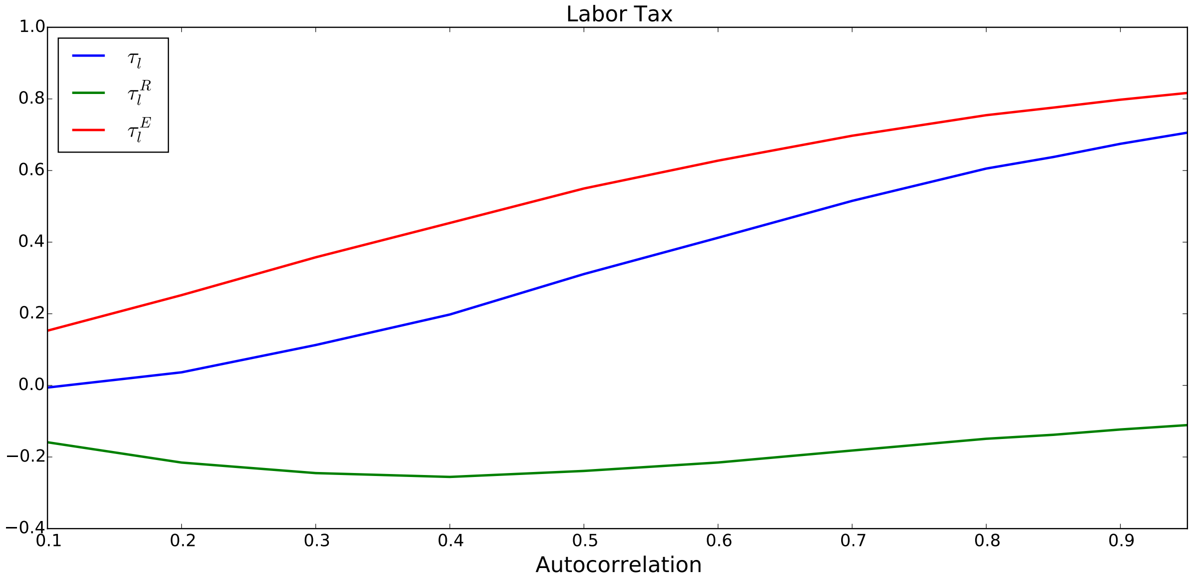

Figure 6: Labor tax after 300 periods decomposed into its two components: $\tau^{l,R}$ and $\tau^{l,E}$

To understand what is driving $\tau^{l,E}$, we first note that in the long run each agent has bond holdings that were purchased in the previous period. Some entrepreneurs have had successful businesses, and thus, have a lot of assets, while others have businesses that have underperformed and are now in debt. In making their labor supply decision each agent is balancing the disutility of supplying labor with the marginal benefit of additional consumption

$$-U^i_{l,t} = (1 - \tau^l_t) W_t U^i_{c,t}$$

The agents do not, however, take into account that if everyone supplies additional labor that increases aggregate consumption and decreases the bond price the previous period. Agents in debt then have to pay a greater portion of the labor income in interest, while those with assets receive additional income from their assets. Thus, the agents who are in debt are not consuming the full amount $(1-\tau_t^l)W_t$ if all agents increases their labor supply uniformly. While the agents do not take this into account, the planner does. Realizing that the poor agents are effectively working too hard, the planner moves to raise labor taxes to reduce their labor supply. In the end, the planner is balancing the two terms $\tau^{l,R}$ and $\tau^{l,E}$ and, as the autocorrelation of the productivity process increases, the increase in inequality causes $\tau^{l,E}$ to dominate, and thus, the planner institutes a positive labor tax.

6 Aggregate Shocks

The flexibility of our solution algorithm allows us to easily incorporate aggregate shocks. As such, we introduce an aggregate component to productivity

$$\log(A_t^i) = \Theta_t + \nu_{a,t}^i + \epsilon_t^i$$

where Θ_t is common to all agents and i.i.d. normal over time with standard deviation 0.03. We can then determine how the optimal Ramsey allocation responds to aggregate shocks to TFP. We are interested in comparing two different cases. The first case maintains the assumption of the previous sections requiring that agents trade only a risk free bond. In the second case we allow the agents to hedge against aggregate shocks by trading a complete set of one period Arrow securities with respect to the aggregate shocks. This allows us to contrast our results to Chari et al. (1994) and Farhi (2010) which studied the optimal Ramsey allocation in a standard neoclassical with complete and incomplete markets respectively. In those models the introduction of incomplete markets dramatically increased the volatility of capital taxes in the optimal Ramsey allocation. With heterogeneous agents we find the opposite result, that the presence of a complete set of Arrow securities with respect to the aggregate shock increases the volatility of capital taxes. As in Farhi (2010) we make the assumption that capital taxes are slow, and must be chosen one period in advance. For the complete markets problem this is equivalent to studying the ex-ante wedge in Chari et al. (1994).

6.1 Results

In order to focus on the contribution of aggregate risk we use the following procedure. We begin by simulating the economy with idiosyncratic investment risk for 300 periods using the calibration in the previous section and an autocorrelation of 0.8. After 300 periods we focus on the limit where the size of the idiosyncratic shocks is zero and the persistence of the idiosyncratic component approaches 1. The economy is then populated by continuum of agents with permanent types. This has the advantage of maintaining heterogeneity across agents, while allowing us to focus entirely on the aggregate risk. We can then study how the Ramsey planner uses capital taxes to respond to aggregate shocks in the presence of concerns for redistribution.

We then perform the experiment of hitting the economy with 1 standard deviation decrease in TFP at period 1 which is then followed by a one standard deviation increase in TFP at period 9. The impulse response of the allocations to this shock is plotted in Figure 7. The blue line is the optimal allocation with complete markets with respect to the aggregate shocks, while the green line represents the optimal allocation when markets are incomplete. What immediately stands out from this plot is that the response of capital taxes flips depending on market structure.

With complete markets, the time series for capital taxes is not surprising. As we are studying a problem with heterogeneous agents, the Ramsey planner has concerns for redistribution. As such, the ergodic level of the capital tax is positive. When the government enters a recession, it becomes optimal for the Ramsey planner to subsidize capital, and hence, reduce the distortions preventing capital accumulation. This result flips, if we initialize the problem with a distribution of states such that planner would want to subsidize capital in the long run.

With incomplete markets, the planner responds in the opposite manner. The reason for this is similar to the forces at work in Farhi (2010). There, if the government enters the period with debt and receives a negative shock to productivity, the Ramsey planner partly finances the shortfall in revenue by taxing capital. This benefits the government two ways: first it raises income from capital taxes, and second it lowers the interest rate reducing the value the inherited

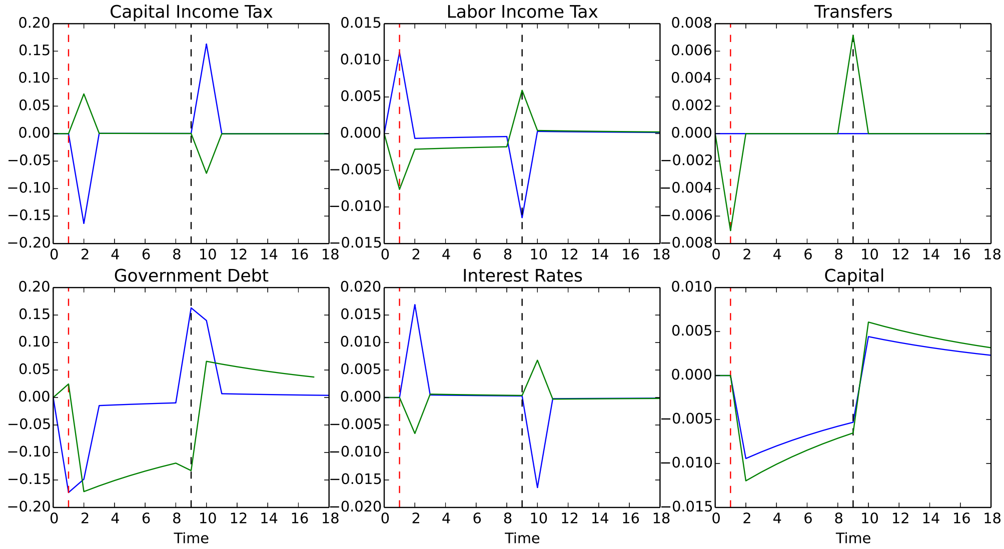

Figure 7: Plots the impulse response to a period of low TFP at time 1 and high TFP at time 2. The blue line represents the response of the allocations under complete markets while the green line is the response with incomplete markets.

debt. As our planner has access to lump sum transfers, it need not worry about financing debt, but it does care about the distribution of consumption.

Figure 8 plots the change in log consumption as function of log consumption prior to the recession when markets are incomplete. We see that, by raising capital taxes, the planner is able to reduce inequality across agents. This in turn, from our previous analysis of the labor tax rate, allows the planner to permanently lower labor taxes. The planner is able to do this precisely because the agents cannot use the asset market to bet against the planners decisions. As the agents only trade a risk free bond, the planner must balance the increase of capital taxes in a recession with a corresponding decrease in capital taxes during a boom. The motivation, however, is clear. When aggregate consumption is low the planner raises taxes to reduce inequality, and when aggregate consumption is high the planner lowers taxes to increase inequality.

The efficiency motive from the complete markets problem is still present, and these two forces push in opposite directions. In this case, they partially balance each other out, and we find that capital taxes are less volatile with incomplete markets. We see this in Table 1 which displays

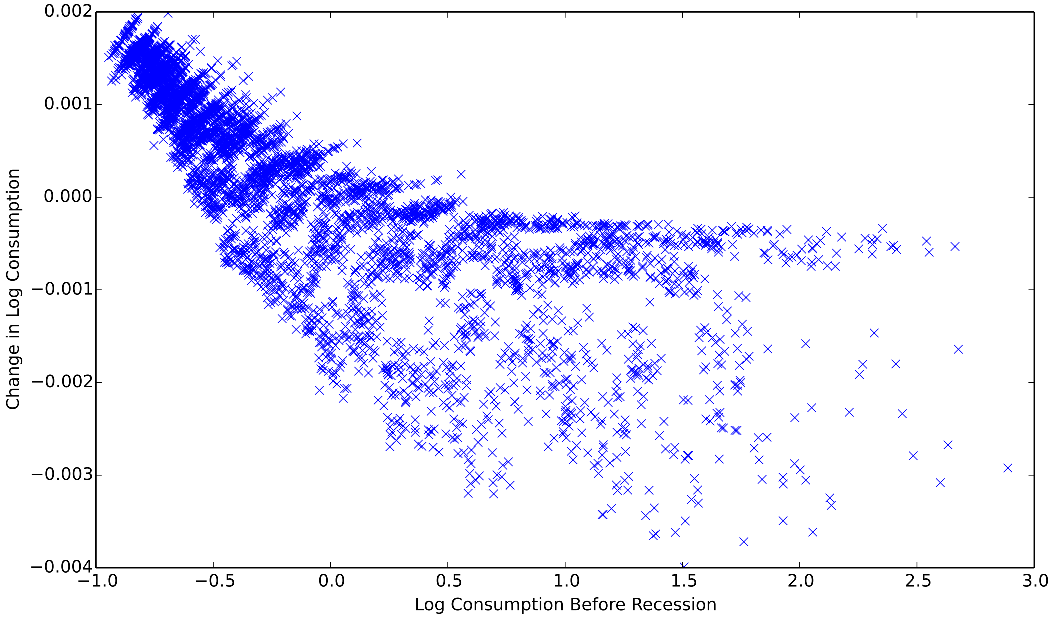

Figure 8: Change in log consumption as a function of consumption prior to the recession.

the moments of the optimal tax policy for a simulation of 400 periods. The standard deviation of the optimal capital is 3 times as large in complete markets as opposed to incomplete markets. As noted before, this is the reverse of what we observe with a representative agent. The overall variation of the labor tax rate remains constant. With complete markets the labor tax rate is essentially i.i.d., but with incomplete markets the labor income tax becomes persistent. This is a direct implication of our analysis above. We saw that income taxes respond to inequality and with incomplete markets the labor income tax inherits the persistent changes to the distribution of consumption.

	Complete Markets		Incomplete Markets	
	Capital Tax	Labor Tax	Capital Tax	Labor Tax
Mean	34%	61%	34%	62%
Standard Deviation	18%	1%	6.4%	1%
Autocorrelation	0.019	-0.03	-0.06	0.6

Table 1: Tax moments of the Ramsey allocation for both cases: complete and incomplete markets.

7 Conclusion

This paper solves a Ramsey taxation problem with uninsurable investment risk in the form of persistent shocks to the productivity of firms run by entrepreneurs. When choosing the optimal capital tax the government is faced with two conflicting motives. On the one hand the presence of investment risk drives the agents to under-invest, which without capital subsidies would result in the capital level being below first best. On the other hand investment risk increases inequality, which the government would want to insure against using higher capital taxes.

In Panousi and Reis (2014) the first motive dominates in the long run, and the government moves to correct the savings decisions as much as possible. This leads the government to subsidize capital in the long run if the investment risk is small enough. We recover this result when the productivity process is i.i.d. and the period length is small (when the discount factor β is close to one). However, as the period length increases the ability of entrepreneurs to smooth consumption, using the risk free asset decreases, providing a greater role for the redistributive role of taxation which eventually dominates.

There is a direct analog between increasing the length of the period and the introduction of persistent productivity shocks. The persistence of productivity shocks links periods together. A higher capital tax in a given period helps insure agents against productivity in the current period, but also redistributes income from productivity shocks arising previous periods. As such, we find the long run capital tax is steadily increasing in the persistence of the productivity process holding constant the cross-sectional distribution of productivities. Consequently, we find the the long run capital tax is increasing in the autocorrelation of the idiosyncratic productivity process of the firms. This increases is almost entirely driven by the redistributive motive of the planner.

It is also necessary to consider heterogeneity when considering how taxes should evolve over the business cycle. In representative agent models the presence of incomplete markets increased the volatility of ex-ante capital taxes. We find the reverse. In our model with heterogeneous agents and a redistributionary motive for capital taxation the volatility of capital taxes is less in an economy incomplete markets as opposed to an economy where agents have a complete set of one period Arrow securities with respect to the aggregate shock.

A Extension: Financial Frictions

Currently, if an agent receives a large permanent shock to his firm's productivity he is perfectly able to finance additional capital investment through borrowing. The planner does not have to worry that by extracting resources through capital she may push firms against credit constraints distorting the economy further. It is therefore reasonable to wonder if the high wealth taxes we observe in the model would be robust to the inclusion of financial frictions.

A common approach to introducing financial frictions faced by firms is to impose a collateral constraint, that firms cannot borrow more that a fraction of their current capital holdings

$$b_t^i \geq -\lambda k_t^i.$$

As we are solving this model using a perturbation approach, the inclusion of periodically binding leverage constraints currently can't be solved with this algorithm. To add financial frictions we introduce a continuum of risk-neutral bankers. The bankers take deposits from the agents and pay the global interest rate R_{t-1}. Entrepreneurs can also borrow from the bankers, but if they do the bankers must hire workers to ensure that the loan is repaid. We assume that the number of workers hired by the firm is some convex function $\psi(b_{t-1}^i)$ which has the properties $\psi(b_{t-1}^i) = 0$ if $b_{t-1}^i \geq 0$ and $\psi'(b_{t-1}^i), \psi''(b_{t-1}^i) < 0$ if $b_{t-1}^i < 0$. The financial market is perfectly competitive, so the interest rate faced by agent i is

$$R_{t-1}^i = R_{t-1} - W_t \psi(b_{t-1}^i)/b_{t-1}^i = R_{t-1} + W_t \hat{\psi}(b),$$

where $\hat{\psi}(b) = -\psi(b)/b$. The introduction of this agent-specific interest rate changes a few of the constraints faced by the planner. Specifically the agent's budget constraint, bond pricing constraint, and the labor market clearing condition have to be adjusted as follows

$$\frac{x^i_{t-1}U^i_{c,t}}{\beta\mathbb{E}_{t-1}U^i_{c,t}} + U^i_{c,t}\left[(1-\tau^k_t)\pi(k^i_{t-1}, A^i_t) - R^i_{t-1}k^i_{t-1}\right] + U^i_{c,t}(1-\tau^l_t)W_t e^i_t l^i_t = U^i_{c,t}(c^i_t - T_t) + x^i_t \tag{36}$$

$$\frac{\alpha_t}{\beta m^i_t \mathbb{E}_t U^i_{c,t+1}} = R^i_t \tag{37}$$

$$\int l^i_t - \psi\left(b^i_{t-1}\right) - n^i_t di = 0 \tag{38}$$

For the planning problem we also need to introduce the constraint

$$b^i_{t-1} = \frac{x^i_{t-1}m^i_{t-1}}{\alpha_{t-1}} - k^i_{t-1} \tag{39}$$

with associated multiplier $\beta^{t-1}\lambda^i_{t-1}$. The first order conditions are updated to include the first order condition with respect to b^i_{t-1}:

$$\lambda^i_{t-1} = \mathbb{E}_{t-1}\left[\rho^i_{2,t-1}W_t\hat{\psi}'(b^i_{t-1}) - \beta U^i_{c,t}\mu^i_t\hat{\psi}'(b^i_{t-1})k^i_{t-1} - \psi'(b^i_{t-1})\eta_t\right] \tag{40}$$

The first order conditions with respect to x^i_{t-1}, m^i_t and k^i_{t-1} have to be updated with appropriate terms including λ^i_{t-1}

$$\begin{aligned}
\mathbf{m^i_t}: \quad & U^i_{c,t}\rho^i_{1,t} + R^i_t\frac{\rho^i_{2,t}}{m^i_t} + (1-\tau^k_{t+1})\frac{\rho^i_{3,t}}{m^i_t} + \frac{x^i_t\lambda^i_t}{\alpha_t} = 0 \\
\mathbf{x^i_{t-1}}: \quad & \frac{\mathbb{E}_{t-1}U^i_{c,t}\mu^i_t}{\mathbb{E}_{t-1}U^i_{c,t}} + \frac{m^i_{t-1}\lambda^i_{t-1}}{\alpha_{t-1}} - \mu^i_{t-1} = 0 \\
\mathbf{k^i_{t-1}}: \quad & \mathbb{E}_{t-1}\Bigg[\mu^i_t U^i_{c,t}\left((1-\tau^k_t)r^i_t - R^i_{t-1}\right) + (1-\tau^k_t)\frac{U^i_{c,t}r^i_{k,t}}{\beta\mathbb{E}_{t-1}U^i_{c,t}r^i_t}\rho^i_{3,t-1} \\
& -f^i_{nk}\phi^i_{2,t} - \beta^{-1}\kappa_{t-1} + f^i_{k,t}\xi_t - \lambda^i_{t-1}/\beta\Bigg] = 0
\end{aligned}$$

In addition, the first order condition with respect to c^i_t must include the individual firm specific interest rate R^i_t.

A.1 Transfers

Our previous analysis assumed that the government could freely adjust transfers. Under that assumption, government debt is no longer a state of the planner's problem, all that matters is the relative distribution of assets across individuals. When solving the planner's problem we make the normalization that $T_t = G$ for all times t, but any other normalization is valid. We show this by noting that the first order condition with respect to transfers is

$$\int \mu_t^i U_{c,t}^i di = 0$$

which is automatically satisfied by the first order condition with respect to x_{t-1}^i (in absence of financial frictions)

$$\mu_{t-1}^i = \frac{\mathbb{E}_t \mu_t^i U_{c,t}^i}{\mathbb{E}_{t-1} U_{c,t}^i}$$

when

$$\int \mu_0^i U_{c,0}^i di = 0$$

holds at time 0.

With the inclusion of financial frictions there is an additional term $\frac{m_{t-1}^i \lambda_{t-1}^i}{\alpha_{t-1}}$ in the evolution of the multiplier μ_t^i which implies that

$$\int \mu_t^i U_{c,t}^i di = 0$$

need no longer hold for all times t. Instead, the planner will wish to satisfy this constraint by using transfers to insure that

$$\int \lambda_t^i di = 0$$

for all times t. The planner can do this by constructing a transfer scheme that is decreasing in time. Agents will then use high transfers in the current period to save in order to pay for lower transfers (or lump sum taxes) in future period. In doing this, the planner can insure that no agent is ever borrowing, and thus, the multiplier λ_t^i is zero for all agents for all time.

In this solution, while agents do have positive assets in the current, those assets are earmarked to pay for future lump sum taxes issued by the government. From the perspective of the bankers, given that the government commits to a schedule of future taxes, this should not be economically different than an agent being in debt and the government committing to set $T_t = 0$ for all time. We therefore assume that credit constraints adjust to the sequence of lump sum transfers set by the government and impose the normalization that $T_t = 0$ for all periods t. Now, in the presence of financial frictions the integral

$$\int \mu_t^i U_{c,t}^i di$$

need no longer be zero. In fact, there will be a mapping from this integral to the total debt of the government.

A.2 Solution Algorithm

With the introduction of these financial frictions, it is no longer the case that in the non-stochastic limit there is always a steady state for any distribution of individual states $(m_t^i, \hat{\mu}_t^i, \nu_{a,t}^i, \nu_{e,t}^i)$. We choose to solve this model by allowing the perturbation parameter to index the cost function ψ. As the size of the idiosyncratic shocks approaches zero we assume that the $\psi(b) \to 0$ for all levels of debt b.

One feature of approximating the solution to the planner's problem in this manner is that the financial frictions only truly applies to agents who are currently borrowing (as we approximate the policy rules for each agent and then aggregate them to find the law of motion for the aggregate variables). If instead we had applied the perturbation analysis around a degenerate non-stochastic steady state then the financial frictions would apply after any deviation from the steady state level of debt.

Currently we only perform a first order approximation with respect to the financial frictions, this means that agents do not anticipate the financial constraints relaxing in the future as they save. Also agents who currently have a positive asset position do not anticipate the increased interest rates they will face if a specific sequence of shocks requires that they borrow in the future. The fact that these effects are not taken into account is unfortunate, but I am currently working on higher order approximations which will allow agents to anticipate how their decisions will effect the future interest rates they face.

A.3 Results

In this section we explore how the results change when ψ is given by

$$\psi(b) = 0.001b^3.$$

In Figure 9 we plot the pdf of bond holdings after 200 periods when the persistence of the productivity process is 0.6. We see that the proportion of agents in debt has decreased while those with slightly positive bond holdings has increased. At the same time the tail of the distribution has remained unchanged. As the net bond holdings of the individual agents have to equal the bond holdings of the government, we must have that the government is issuing more debt when facing financial frictions, but in issuing more debt the planner must also raise taxes.

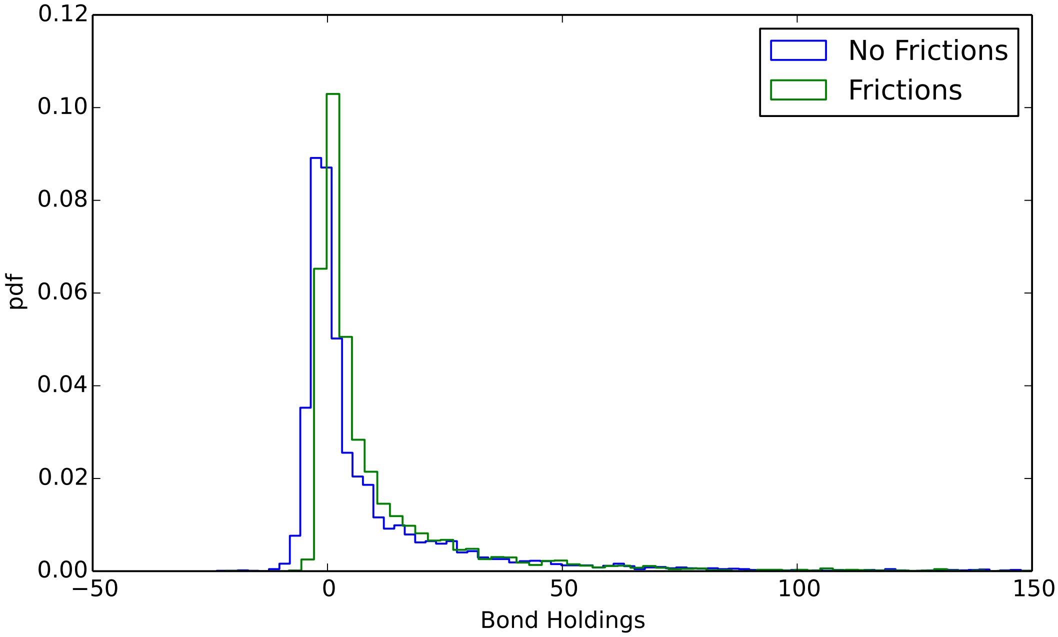

Figure 9: The distribution of bond holdings after 200 period for the economy with (green) and without (blue) financial frictions.

In Figure 10 we plot the long run tax rates for both the case with and without financial frictions as a function of the persistence of the productivity process. We see that the planner

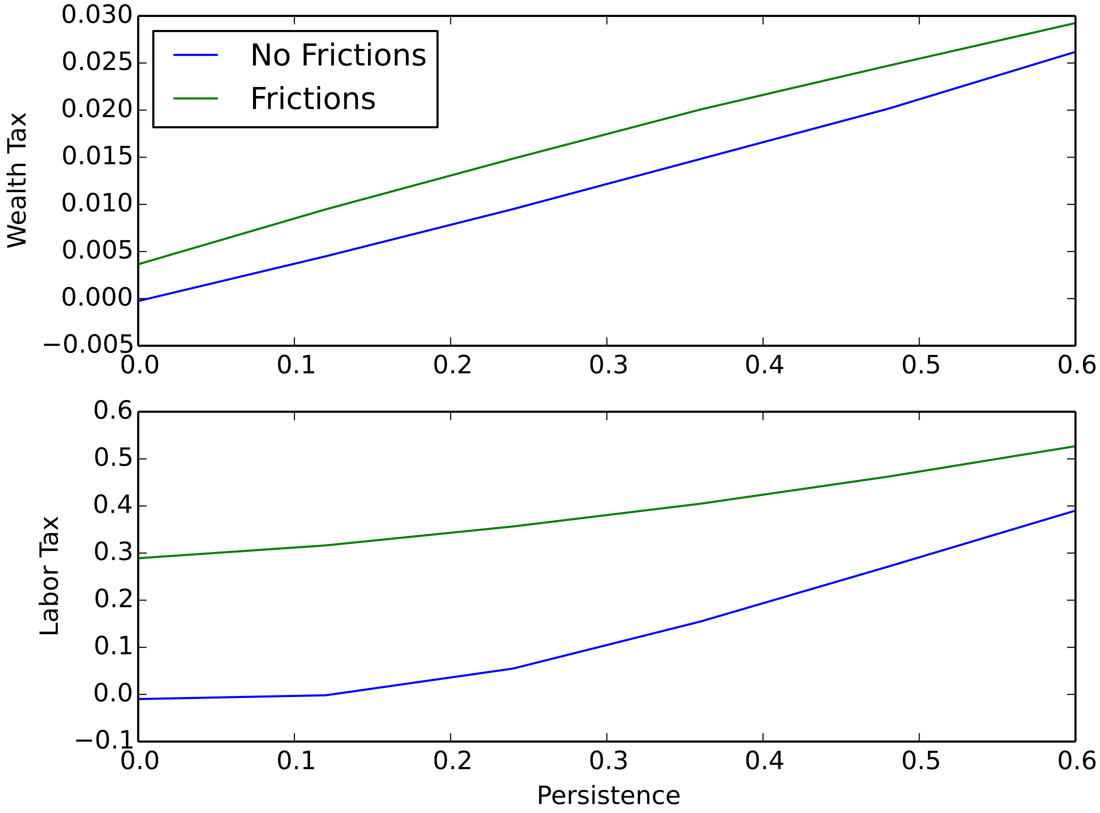

Figure 10: The tax rate after 200 periods as a function of the persistence of the productivity process when agents do (green) and do not (blue) face financial frictions.

does raise capital taxes uniformly across the different values of persistence. The introduction of financial frictions means that

$$\int \mu_t^i U_{c,t}^i di$$

need no longer be zero, in fact, will become negative representing the government need to extract resources to cover the debt that it issues. As the entrepreneurs operate decreasing returns to scale production functions the government will want to extract rents by raising the capital tax. Most of the financing, however, comes from the labor tax. The government chooses to increase saving by the entrepreneurs by issuing more debt and raising future taxes, rather than subsidizing investment.

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